Multiscale analyses for the Shallow Water equations

Didier Bresch, Rupert Klein, and Carine Lucas

Abstract This paper explores several asymptotic limit regimes for shallow water flows over multiscale topography. Depending on the length and time scales considered and on the characteristic water depth and height of topography, a variety of mathematically quite different asymptotic limit systems emerges. Specifically, we recover the classical "lake equations" for balanced flow without gravity waves in the single time, single space scale limit (Greenspan, Cambridge Univ. Press, (1968)), discuss a weakly nonlinear and a strongly nonlinear multi-scale version of these wave-free equations involving short-range topography, and we re-derive the equations for long-wave shallow water waves passing over short-range topography by Le Maître et al., JCP (2001).

1 Introduction

1.1 Governing equations and non-dimensionalization

In this article, we present multiscale analyses for the shallow water equations,

$$\begin{aligned} \partial_t H + \operatorname{div}(Hu) &= 0 \\ \partial_t (Hu) + \operatorname{div}(Hu \otimes u) + g\nabla H^2/2 &= -gH\nabla b \end{aligned}, \tag{1}$$

Didier Bresch
Université de Savoye, Chambéry, e-mail: Didier.Bresch@univ-savoie.fr

Rupert Klein
Freie Universität Berlin e-mail: Rupert.Klein@math.fu-berlin.de

Carine Lucas
Universit'e de Orleans e-mail: carine.lucas@univ-orleans.fr

where $(H,u)(t,x)$ are the depth of the water layer and the flow velocity, g is the gravitational acceleration, $b(x)$ denotes the bottom topography, (t,x) are time and horizontal space coordinates, and $\otimes$ indicates the tensorial product. Non-dimensionalizing (H,u,b,x,t) by characteristic values $H_{\text{ref}}, u_{\text{ref}}, b_{\text{ref}}, L, t_{\text{ref}}$, respectively, we obtain, using the same symbols for the dimensionless variables as used above for the dimensional ones,

$$
\begin{aligned}
\mathrm{Sr}\,\partial_t H + \mathrm{div}(Hu) &= 0 \\
\mathrm{Sr}\,\partial_t(Hu) + \mathrm{div}\,(Hu\otimes u) + \frac{1}{\mathrm{Fr}^2}\nabla H^2/2 &= -\frac{\beta}{\mathrm{Fr}^2}H\nabla b
\end{aligned}
\,. \tag{2}
$$

Here

$$
\mathrm{Sr} = \frac{L}{t_{\text{ref}}u_{\text{ref}}}\,, \qquad \mathrm{Fr} = \sqrt{\frac{gH_{\text{ref}}}{u_{\text{ref}}^2}}\,, \qquad \beta = \frac{b_{\text{ref}}}{H_{\text{ref}}}\,, \tag{3}
$$

are the Strouhal, and Froude numbers, and the ratio of typical variations of the topography versus the water layer depth.

The asymptotic limit regimes to be analyzed in this paper will be defined by particular distinguished limits of these dimensionless parameters, and by multiple spacio-temporal scales. Throughout, we consider low Froude number flows, for which flow velocities are systematically small compared with the speed of the gravity waves, and we introduce the reference asymptotic expansion parameter, ε, via

$$
\mathrm{Fr} = \varepsilon^{\alpha} \ll 1\,, \tag{4}
$$

with α depending on the particular flow regime considered.

As discussed in [8], low Froude numbers give rise to multiple length or time scales or both, depending on the particular set-up of initial and boundary conditions, and on the structure of any pertinent source terms. For example, a typical distance $t_{\text{ref}}u_{\text{ref}} = L/\mathrm{Sr}$ which an advected particle traverses during the reference time t_{ref} differs asymptotically from the distance $t_{\text{ref}}\sqrt{gH_{\text{ref}}} = (L/\mathrm{Sr})/\varepsilon^{\alpha}$ traversed by a shallow water gravity during the same time. Similarly, variations of an advected variable over length scales of order $O(L)$ induce temporal variations of that quantity on a time scale $L/u_{\text{ref}} = \mathrm{Sr}\,t_{\text{ref}}$, whereas gravity waves with characteristic length L feature asymptotically shorter time scales of order $L/\sqrt{H_{\text{ref}}} = \varepsilon^{\alpha}\,\mathrm{Sr}\,t_{\text{ref}}$. See also the analogous discussion of low Mach number compressible flows in [7].

We are interested here in multiscale topography. The relevant scalings as adopted in the asymptotic analyses to follow below are indicated in Fig. 1. Thus, for $\varepsilon \ll 1$, we refer to a "normal scale", resolved by a dimensionless coordinate x, a short range resolved by $X = x/\varepsilon$, and a long-wave scale represented by $\chi = \varepsilon x$. Depending on the flow regime considered, ε is representative of different powers of the Froude and Strouhal numbers as discussed shortly.

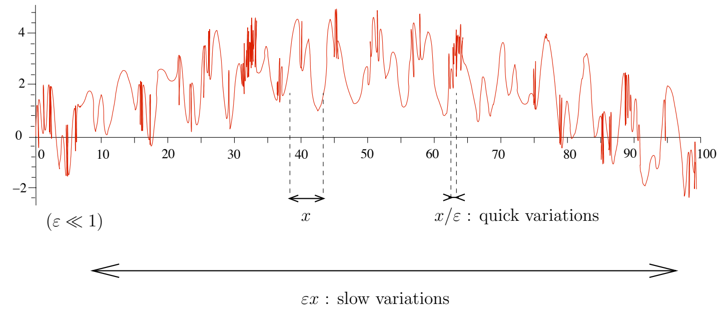

Fig. 1 Slow, classical and fast scales.

1.2 Asymptotic flow regimes and organization of the paper

Essentially different flow characteristics emerge under the following different distinguished asymptotic limits, all of which we will analyse in some detail below. We will assume

$$\beta \equiv 1 \tag{5}$$

throughout, i.e., we allow for topographical heights comparable with the shallow water depth, and otherwise consider:

- **The "Lake Equations": single-scale, inviscid balanced flow over topography** (Section 2)

  We let $\mathrm{Fr} = \varepsilon$, i.e., $\alpha = 1$, follow the flow over advective time scales so that $\mathrm{Sr} = 1$, and assume

$$\left.\begin{array}{l}\mathrm{Fr} = \varepsilon\\ \mathrm{Sr} = 1\end{array}\right\} \quad (H,u)(t,x;\varepsilon) = \sum_i \varepsilon^i \, (H,u)^i \, (t,x)\,, \quad b(x;\varepsilon) \equiv B(x)\,. \tag{6}$$

- **Inviscid balanced flow over multiscale topography** (Section 3)

  Similar to the previous case, we consider multiscale topography, yet this time we assume its second characteristic scale to be much shorter than L instead of much longer. At the same time, by letting $\mathrm{Fr}(\varepsilon) = o(\mathrm{Sr}(\varepsilon))$ as $\varepsilon \to 0$, we restrict the considered flows to characteristic lengths that are too short to support gravity waves over the given time scales. Moreover, we distinguish a weakly nonlinear and a fully nonlinear regime, characterized by

  weakly nonlinear regime (Section 3.1)

$$\left.\begin{array}{r}\mathrm{Fr}^2 = \varepsilon^3\\ \mathrm{Sr} = \varepsilon^{-1}\end{array}\right\} \quad (H,u)(t,x;\varepsilon) = \sum_i \varepsilon^i \, (H,u)^i \left(t, \frac{x}{\varepsilon}, x\right)\,; \quad b(x;\varepsilon) = B\left(\frac{x}{\varepsilon}, x\right) \tag{7}$$

  fully nonlinear regime: (Section 3.2)

$$\left.\begin{array}{l}\mathrm{Fr}=\varepsilon\\ \mathrm{Sr}=1\end{array}\right\} \quad (H,u)(t,x;\varepsilon)=\sum_i \varepsilon^i\,(H,u)^i\left(t,\frac{x}{\varepsilon},x\right)\,;\;\; b(x;\varepsilon)=B\left(\frac{x}{\varepsilon},x\right). \tag{8}$$

In these balanced, i.e., waveless, flow regimes, gradients of the water height on scale L drive mean flows, which in conjunction with the short-range topography induce a small-scale flow response. Depending on whether we adopt the scalings for strong or weak nonlinearity, the nonlinear advection of momentum by the small-scale flow does or does not affect in turn the mean flow dynamics at leading order, respectively. In case of strong nonlinearity, the result is a nonlinear Darcy-type homogenized equation system, with homogenized nonlinear advective momentum transport replacing the viscous fluxes in the classical Darcy theory.

- **Gravity waves over multiscale topography** (Section 4)

  Again we let $\mathrm{Fr}=\varepsilon$ and $\mathrm{Sr}=1$, but express $(H,u)(t,x;\varepsilon)$ and $b(x;\varepsilon)$ through

$$(H,u)(t,x;\varepsilon)=\sum_i \varepsilon^i\,(H,u)^i\,(t,x,\varepsilon x)\,,\qquad b(x;\varepsilon)=B(x,\varepsilon x)\,. \tag{9}$$

  In this regime we will observe how long-range gravity waves generate localized balanced flow over the normal-scale topography, and how they will be affected by the induced effective average nonlinear momentum transport. Analogous regimes were studied in [7] for weakly compressible flow with small-scale entropy and vorticity in the context of ocean flows in [9], and for near-equatorial atmospheric motions in [12].

  We draw conclusions in section 5.

2 Single-scale limit: the “lake equations”

Here we let $\mathrm{Fr}=\varepsilon$ and $\mathrm{Sr}=1$, so that the dimensionless equations from (2) become

$$\begin{aligned}\partial_t H+\mathrm{div}(Hu)&=0\\ \partial_t(Hu)+\mathrm{div}(Hu\otimes u)+\frac{H}{\varepsilon^2}\nabla(H+B)&=0\end{aligned}\,. \tag{10}$$

Single-scale expansion of the solution in terms of ε according to

$O(\varepsilon^{-2})$

$$H^0\nabla(H^0+B)=0\,, \tag{11}$$

$O(\varepsilon^{-1})$

$$H^1\nabla(H^0+B)+H^0\nabla H^1=0\,, \tag{12}$$

$O(\varepsilon^0)$

$$\begin{aligned} \partial_t H^0 + \operatorname{div}(H^0 u^0) &= 0 \\ \partial_t (H^0 u^0) + \operatorname{div}(H^0 u^0 \otimes u^0) + H^2 \nabla (H^0 + B) + H^1 \nabla H^1 + H^0 \nabla H^2 &= 0 \end{aligned} . \tag{13}$$

From (11), (12) we conclude that

$$H^0 + B \equiv c^0(t) \tag{14}$$

and $H^1 \equiv c^1(t)$. Then, since the topography, B, is assumed time independent, we have from (13) that

$$\begin{aligned} \operatorname{div}(H^0 u^0) &= -\frac{dc^0}{dt} \\ \partial_t (H^0 u^0) + \operatorname{div}(H^0 u^0 \otimes u^0) + H^0 \nabla H^2 &= 0 \end{aligned} . \tag{15}$$

The time change of the total water height, $c^0(t)$, follows from integrating (15) over the entire flow domain to be

$$\frac{dc^0}{dt} = -\frac{1}{|\Omega|} \int_{\Omega} H^0 u^0 \cdot n \, d\sigma \,, \tag{16}$$

i.e., the change of water height is given by the total flux of water across the domain boundary.

Equations (14)–(16) constitute the classical zero Froude number shallow water or “lake equations”. They form a closed system, once appropriate initial and boundary conditions for u^0, c^0 are provided. See [5].

3 Inviscid balanced flow over short-wave topography

The second choice for the topography is to consider a bottom that depends on x and on the fast variable $X = x/\varepsilon$, such as the one plotted on Figure 2. For this case, we expand the velocity and the water height as

$$\begin{aligned} u(t,x;\varepsilon) &= u^0(t,X,x) + \varepsilon u^1(t,X,x) + \ldots \\ H(t,x;\varepsilon) &= H^0(t,X,x) + \varepsilon H^1(t,X,x) + \ldots \end{aligned} , \qquad X = \frac{x}{\varepsilon} \,, \tag{17}$$

and we replace them in the Shallow Water system from (2), for (X,x) in $\mathbb{T}^2 \times \mathscr{D}$.

We begin in section 3.1 with a study of the weakly nonlinear system that arises under the distinguished limit $\mathrm{Sr} = \varepsilon^{-1}, \mathrm{Fr}^2 = \varepsilon^3$ for the Strouhal and Froude numbers (see (7)): we show that the limit as $\varepsilon \to 0$ leads to a weakly nonlinear limit version of the lake equations with oscillatory topography. In section 3.2, we study the Shallow Water equations with a strong nonlinearity and show that the balanced multiscale regime leads to a new nonlinear Darcy-type problem in which cumulative

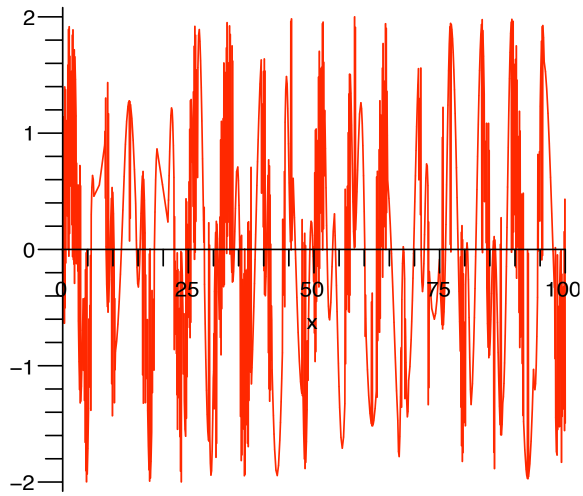

Fig. 2 Example of a topography that depend on the fast variable $X = x/\varepsilon$ and on x.

inertial forces of the small-scale flow through the topography replaces the viscous forces in the classical Darcy model.

3.1 Weakly nonlinear regime

With $\mathrm{Sr} = \varepsilon^{-1}$ and $\mathrm{Fr}^2 = \varepsilon^3$, the dimensionless shallow water equations from (2) read

$$\begin{aligned} \partial_t H + \varepsilon \mathrm{div}(Hu) &= 0 \\ \partial_t (Hu) + \varepsilon \mathrm{div}(Hu \otimes u) + \frac{H}{\varepsilon^2} \nabla (H + b) &= 0 \end{aligned} . \tag{18}$$

Assuming multiscale topography such that $b(x;\varepsilon) \equiv B(x/\varepsilon, x)$, and adopting the asymptotic expansion scheme from (17), we identify terms multiplied by like powers of ε,

$O(\varepsilon^{-3})$

$$H^0 \nabla_X (H^0 + B) = 0, \tag{19}$$

$O(\varepsilon^{-2})$

$$H^0 \nabla_x (H^0 + B) + H^1 \nabla_X (H^0 + B) + H^0 \nabla_X H^1 = 0\,, \tag{20}$$

$O(\varepsilon^{-1})$

$$H^1 \nabla_x (H^0 + B) + H^0 \nabla_x H^1 + H^2 \nabla_X (H^0 + B) + H^1 \nabla_X H^1 + H^0 \nabla_X H^2 = 0\,, \tag{21}$$

$O(\varepsilon^0)$

$$\begin{aligned} \partial_t H^0 + \mathrm{div}_X (H^0 u^0) &= 0 \\ \partial_t (H^0 u^0) + \mathrm{div}_X (H^0 u^0 \otimes u^0) + H^2 \nabla_x (H^0 + B) + H^1 \nabla_x H^1 & \\ + H^0 \nabla_x H^2 + H^3 \nabla_X (H^0 + B) + H^2 \nabla_X H^1 + H^1 \nabla_X H^2 + H^0 \nabla_X H^3 &= 0 \end{aligned} \quad , \tag{22}$$

$O(\varepsilon^1)$

$$\partial_t H^1 + \mathrm{div}_x (H^0 u^0) + \mathrm{div}_X (H^1 u^0) + \mathrm{div}_X (H^0 u^1) = 0. \tag{23}$$

Equation (19) enables us to assert that, at leading order, the water height reads: $H^0(t,X,x) = -B(X,x) + c(t,x)$. Then the second term of (20) is equal to zero and, taking the mean value in X, we find that $H^0(t,X,x) + B(X,x)$ does not depend on x. In the same way with $(22)_1$, we have $\partial_t (H^0 + B) + \mathrm{div}_X (H^0 u^0) = 0$, and, computing the mean value in the fast variable, we get that the function c does not even depend on the time: it is a constant in time and space, given by the value of H^0 at the initial time. Consequently we know that

$$H^0(X,x) \equiv -B(X,x) + C \qquad \text{where} \qquad C = B + H^0{}_{|t=0}\,. \tag{24}$$

With this result and (20), we obtain $\nabla_X H^1 = 0$. Then, (21) integrated in X yields that H^1 does not depend on x, so the first order of the water height does not depend on the space variables either. Averaging (23) in X and integrating in x over the (finite) domain, we find

$$\frac{dH^1}{dt} = -\oint_{\Omega} \overline{H^0 u^0} \cdot n\, d\sigma\,, \tag{25}$$

where here and below, an overbar denotes averaging over the small-scale coordinate.

Equation (25) states that the first-order water height can change in time only due to fluxes through the boundary of the overall flow domain. From here on we assume for simplicity that rigid vertical walls bound the domain, and in that case, $H^1 \equiv$ const., and we may assume $H^1 \equiv 0$. Equipped with this result, the sublinear growth condition applied to (23) provides

$$\mathrm{div}_x \overline{H^0 u^0} = 0\,. \tag{26}$$

Inserting the results obtained thus far in (21) we find $\nabla_X H^2 = 0$, and then (22), (23) yield the following multiscale system of equations for H^2, H^3, and the leading-order velocity, u^0,

$$\begin{aligned} \partial_t (H^0 u^0) + \mathrm{div}_X (H^0 u^0 \otimes u^0) + H^0 \nabla_x H^2 + H^0 \nabla_X H^3 &= 0 \\ \mathrm{div}_X (H^0 u^0) &= 0 \\ \mathrm{div}_x \overline{H^0 u^0} &= 0 \\ \nabla_X H^2 &= 0 \end{aligned} \quad , \tag{27}$$

with $H^0(X,x) = C - B(X,x)$.

We are interested in separately extracting the large and small-scale dynamics represented by this system. Averaging all equations in X eliminates the second and fourth equations, whereas the first and third become the governing equations for the

Non-stationary linear balanced mean flow

$$\begin{aligned} \partial_t \overline{H^0 u^0} + \overline{H^0} \nabla_x H^2 &= -\overline{H^3 \nabla_X B} \\ \operatorname{div}_x \overline{H^0 u^0} &= 0 \end{aligned} \quad . \tag{28}$$

This equation describes how the leading-order large-scale flow responds to accumulated small-scale pressure forces on the topography, with its divergence constraint enforced by adjustment of the second order height, H^2, which acts as the usual Lagrangian multiplier.

Subtracting (28) from (27), and observing that $H^0 + B \equiv C$ implies $H^0 - \overline{H^0} \equiv \widetilde{H^0} = -\widetilde{B}$, we obtain the governing equations for the

Non-stationary nonlinear balanced small-scale flow

$$\begin{aligned} \partial_t \widetilde{H^0 u^0} + \operatorname{div}_X (H^0 u^0 \otimes u^0) + \widetilde{H^0 \nabla_X H^3} &= \widetilde{B} \nabla_x H^2 \\ \operatorname{div}_X \widetilde{H^0 u^0} &= 0 \end{aligned} \quad . \tag{29}$$

Here and below, the tildae denote small-scale fluctuations.

Equations (28) and (29) reveal how the small and large-scale flow components interact through their respective Lagrangian multiplier pressure fields. The gradient of the second-order height H^2, which is equivalent to a pressure field and purely large-scale, acts on on fluctuations of the topography to drive the small-scale flow. The divergence of the latter is controlled by the third-order height (or pressure) field, H^3. This field produces an accumulated topographical force, $\overline{H^3 \nabla_X B}$, which in turn drives the large-scale flow.

Energy conservation

The multi-scale system derived here observes an energy principle. Multiplying Equation $(27)_1$ by u^0, and integrating in x and X by parts we get

$$\frac{1}{2}\frac{d}{dt}\int\int H^0 \left|u^0\right|^2 dx\,dX - \int\int \left(H^2 \operatorname{div}_x (H^0 u^0) + H^3 \operatorname{div}_X (H^0 u^0)\right) dx\,dX = 0\,. \tag{30}$$

The last term is equal to zero thanks to $(27)_2$, whereas the integrand of the second term may be rewritten as $-H^2 \left(\operatorname{div}_X (H^0 u^1)\right)$, thanks to Equation (23), and the previous conclusion that $H^1 \equiv 0$. The requirement of sublinear growth of $H^0 u^1$ in X then eliminates this term, and we find

$$\frac{1}{2}\frac{d}{dt}\int\int H^0 \left|u^0\right|^2 dx\,dX = 0. \tag{31}$$

The system in (27) or (28) combined with (29) are obtained here through multiple-scales asymptotics from the full "compressible" Shallow Water equations under a particular distinguished limit of the Strouhal and Froude numbers, and of the asymptotic separation of the topographic scales. In contrast, the same system was obtained by Bresch and Varet in [3] through a sequential limit of two small parameters: first they let the Froude number vanish to obtain the balanced "lake equations", and then adopt a distinguished limit between the Strouhal number (their parameter η), and the ratio of the characteristic spacial scales. In this setting, Bresch and Varet prove convergence to the limit model with the two-scale method, used, e.g., in [10] for homogenization problems.

3.2 Fully nonlinear regime

We are now interested in a strong nonlinear term, that is, we adopt order one Strouhal number, and furthermore we let $\mathrm{Fr} = \varepsilon$. Thus we consider the Shallow Water equations in the form

$$\begin{aligned} \partial_t H + \operatorname{div}(Hu) &= 0 \\ \partial_t (Hu) + \operatorname{div}(Hu \otimes u) + \frac{H}{\varepsilon^2} \nabla (H + b) &= 0 \end{aligned}, \tag{32}$$

with multiscale topography, so that $b(x;\varepsilon) = B(x/\varepsilon, x)$.

We perform the same expansion for our variables as in the weakly nonlinear case, see (17), and we identify like powers of ε in the expanded equations,

$O(\varepsilon^{-3})$

$$H^0 \nabla_X (H^0 + B) = 0\,, \tag{33}$$

$O(\varepsilon^{-2})$

$$H^0 \nabla_x (H^0 + B) + H^1 \nabla_X (H^0 + B) + H^0 \nabla_X H^1 = 0\,, \tag{34}$$

$O(\varepsilon^{-1})$

$$\begin{aligned} \operatorname{div}_X (H^0 u^0) &= 0 \\ \operatorname{div}_X (H^0 u^0 \otimes u^0) + H^1 \nabla_x (H^0 + B) + H^0 \nabla_x H^1 + H^2 \nabla_X (H^0 + B) & \\ + H^1 \nabla_X H^1 + H^0 \nabla_X H^2 &= 0 \end{aligned}\,, \tag{35}$$

$O(\varepsilon^0)$

$$\partial_t H^0 + \operatorname{div}_x (H^0 u^0) + \operatorname{div}_X (H^0 u^1) + \operatorname{div}_X (H^1 u^0) = 0\,, \tag{36}$$

3.2.1 Sublinear growth conditions

Here we derive the effective balanced flow equations for the leading-order solutions from (33)–(36). First, (33) yields

$$\nabla_X (H^0 + B) \equiv 0 \tag{37}$$

as in the weakly nonlinear case. Using this information in (34), we obtain after division by H^0 and averaging in X that

$$\nabla_x (H^0 + B) = \nabla_X H^1 \equiv 0\,. \tag{38}$$

For convenience, we rewrite (35) here taking into account the results just obtained, viz.

$$\mathrm{div}_X (H^0 u^0) = 0\,, \tag{39}$$

$$\mathrm{div}_X (H^0 u^0 \otimes u^0) + H^0 \nabla_x H^1 + H^0 \nabla_X H^2 = 0\,. \tag{40}$$

Equation (40) lends itself to two independent sublinear growth constraints. To obtain the first we simply average in X, to obtain the second we divide by H^0 and then average in X. This yields

$$\overline{H^0}\,\nabla_x H^1 + \overline{H^0 \nabla_X H^2} = 0\,, \tag{41}$$

$$\overline{u^0 \cdot \nabla_X u^0} + \nabla_x H^1 = 0\,. \tag{42}$$

3.2.2 Equations for the small-scale flow

Using (39), we conclude from (40) that

$$u^0 \cdot \nabla_X u^0 + \nabla_X H^2 = -\nabla_x H^1\,. \tag{43}$$

We eliminate H^2 from this equation by taking the curl, using $u \cdot \nabla u = \nabla u^2/2 - u \times (\nabla \times u)$, and, using $\omega = \nabla \times u$ with $\mathrm{div}\,\omega \equiv 0$, we have $\nabla \times (u \times \omega) = \omega \cdot \nabla u - \omega\,\mathrm{div}\,u - u \cdot \nabla \omega$ and, taking into account that we have a two-dimensional flow only,

$$u^0 \cdot \nabla_X \zeta^0 + \zeta^0 \mathrm{div}_X u^0 = \mathrm{div}_X (\zeta^0 u^0) = 0\,, \tag{44}$$

where $\zeta = k \cdot \omega = -\partial_{X_2} u_1 + \partial_{X_1} u_2$ and k is the unit vector normal to the flow plane. Combining this result with the divergence condition in (39), we find

$$H^0 u^0 \cdot \nabla_X (\zeta^0 / H^0) = 0\,, \tag{45}$$

so that

$$\zeta^0 = H^0 Q(\psi^{*,0}, x, t)\,, \tag{46}$$

where $\psi^{*,0}$ is a stream function for the mass flux $H^0 u^0$ satisfying

$$\psi^{*,0} = \psi^0 + X^\perp \cdot \overline{H^0 u^0} \qquad \text{with} \qquad H^0 u^0 = \nabla_X^\perp \psi^{*,0}\,, \tag{47}$$

and $Q(\psi,x,t)$ is a prescribed potential vorticity distribution function. Collecting, we obtain an elliptic determining equation for the fluctuating part of the stream function, ψ^0,

$$H^0\nabla_X^2\psi^0 - \nabla_X H^0\cdot\nabla_X\psi^0 = (H^0)^3 Q(\psi^{*,0},x,t) - \nabla_X H^0\cdot\overline{H^0u^0}^{\perp}. \tag{48}$$

This is the cell problem for a stationary vortical flow over variable topography with prescribed vorticity on each of the stream surfaces and with a prescribed large-scale mass flow.

3.2.3 Equations for the large-scale flow

In the previous section we have determined the leading-order velocity, u^0, only up to a large-scale mean mass flux, $\overline{H^0u^0}$. We obtain its governing equation by explicitly introducing the split of the mass flux into mean and fluctuations, i.e.,

$$H^0u^0 = U + H^0\widetilde{u} \quad \text{where} \quad U = \overline{H^0u^0}, \quad \widetilde{u} = u^0 - \frac{1}{H^0}\overline{H^0u^0} = \frac{1}{H^0}\nabla_X^{\perp}\psi^0, \tag{49}$$

and averaging (43) in X. This procedure yields

$$U\cdot T + \nabla_x H^1 = -q \tag{50}$$

where

$$T = \overline{\frac{1}{H^0}\nabla_X\widetilde{u}} \qquad \text{and} \qquad q = \overline{\widetilde{u}\cdot\nabla_X\widetilde{u}}. \tag{51}$$

This is a nonlinear Darcy-type problem, with the effective mean friction tensor, T, and q the accumulated inertial force from the small-scale flow. The latter being determined by the nonlinear cell problem defined in the previous section.

A determining equation for the first-order pressure is obtained by averaging (36) in X, so that

$$\mathrm{div}_x U = -\mathrm{div}_x\left((\nabla_x H^1 + q)\cdot T^{-1}\right) = -\frac{d\overline{H}^0}{dt}. \tag{52}$$

Equations (48)–(52) define an interesting stationary multi-scale problem which, to the best of our knowledge, has not been derived or studied before. A large-scale mean mass flux drives a quasistationary small-scale flow over the topography. The small-scale dynamics is determined by vorticity transport, especially vortex stretching do to the motion in a layer with variable height. The large-scale flow adjusts, in turn, to two accumulated forcings from the small scales both of which are induced by nonlinear advection of momentum. The first results from the advection of small-scale momentum by the mean flow (first term in (50)), whereas the second is the nonlinear average of the nonlinear self-advection of the small-scale momentum (right-hand side of (50)).

4 Gravity waves over long-wave modulated topography

Here we let $\mathrm{Sr} = 1$ to focus on advective times for the normal scale L, and $\mathrm{Fr} = \varepsilon$ so as to include gravity wave dynamics on a large scale L/ε. We start again from the dimensionless shallow water equations in (10), and we are now interested in large flow domains with characteristic extension $O(L/\varepsilon)$, and we allow for multi-scale bottom topography with associated long-wave modulations, so that $b(x;\varepsilon) = B(x, \varepsilon x)$, see Fig. 3. These scalings are analogous to those considered in [7] and [9] for weakly compressible flows of a gas and for oceanic motions, respectively. These authors pursued asymptotic analyses to motivate related specialized numerical schemes.

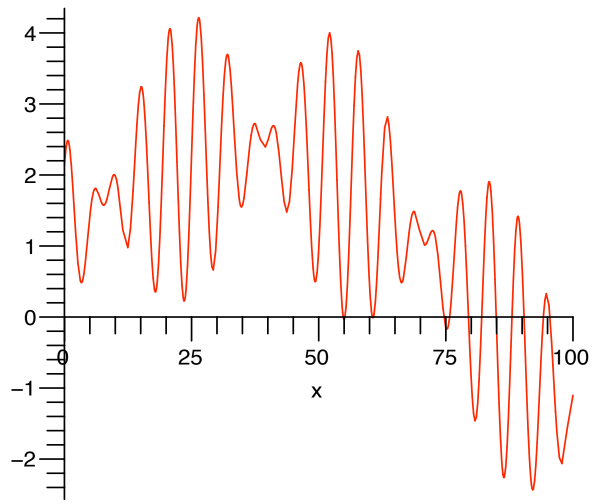

Fig. 3 Example of multiscale topography with leading-order variations on the normal scale represented by the x-coordinate, and long-wave modulations resolved by the slow variable $\chi = \varepsilon x$.

We expand the flow variables as

$$(H,u)(t,x;\varepsilon) = (H,u)^0(t,x,\chi) + \varepsilon\,(H,u)^1(t,x,\chi) + \ldots, \qquad (\chi = \varepsilon x) \tag{53}$$

and, after inserting into (10) collect terms involving like powers of ε as usual,

$O(\varepsilon^{-2})$

$$H^0 \nabla_x (H^0 + B) = 0 \tag{54}$$

$O(\varepsilon^{-1})$

$$H^0 \nabla_x H^1 + H^0 \nabla_\chi (H^0 + B) + H^1 \nabla_x (H^0 + B) = 0 \tag{55}$$

$O(\varepsilon^0)$

$$\partial_t H^0 + \mathrm{div}_x (H^0 u^0) = 0, \tag{56}$$

$$\partial_t (H^0 u^0) + \mathrm{div}_x (H^0 u^0 \otimes u^0) + H^0 \nabla_x H^2 \tag{57}$$

$$+H^1\nabla_x H^1 + H^2\nabla_x(H^0+B) + H^0\nabla_\chi H^1 + H^1\nabla_x(H^0+B) \,=\, 0\,,$$

$O(\varepsilon)$

$$\partial_t H^1 + \mathrm{div}_x(H^0u^1) + \mathrm{div}_x(H^1u^0) + \mathrm{div}_\chi(H^0u^0) = 0\,. \tag{58}$$

Equation (54) requires H^0+B not to depend on x. Integrating (55) in x, after division by H^0, we find H^0+B to be independent of χ as well through a sublinear growth (or secular) condition, so that the water surface is flat to leading order. Inserting this result back, we conclude that H^1 is independent of x as well.

Next we observe that we may replace $\partial_t H^0 \equiv \partial_t(H^0+B)$ in (56), as the bottom topography is supposed to be time independent. Then, integrating again in x, we find as a sublinear growth condition that H^0+B is time independent. This also implies $\partial_t H^0 \equiv 0$ and we conclude that

$$\mathrm{div}_x(H^0u^0) = 0\,, \tag{59}$$

i.e., that the leading-order small-scale flow is "incompressible". Equations (57)–(58) then become

$$\partial_t(H^0u^0) + \mathrm{div}_x(H^0u^0\otimes u^0) + H^0\nabla_x H^2 + H^0\nabla_\chi H^1 \,=\, 0, \tag{60}$$

$$\partial_t H^1 + \mathrm{div}_x(H^0u^1) + \mathrm{div}_x(H^1u^0) + \mathrm{div}_\chi(H^0u^0) \,=\, 0. \tag{61}$$

This is a multiscale system, which we decompose by averaging in x and subtracting the result from the above into separate but coupled descriptions of the long-wave and short-wave components of the flow. Thus we obtain the

Long-wave equations for rough topography

$$\partial_t\left(\overline{H^0u^0}\right) + \overline{H^0}\nabla_\chi H^1 \,=\, \overline{H^2\nabla_x H^0}\,, \tag{62}$$

$$\partial_t H^1 + \mathrm{div}_\chi\left(\overline{H^0u^0}\right) \,=\, 0\,. \tag{63}$$

These are the standard linearized shallow water wave equations, except for the momentum source term on the r.h.s. of (62) which we have derived integrating $H^0\nabla_x H^2$ by parts and using a sublinear growth constraint to eliminate the arising divergence term. This source term represents the net resistance, or large-scale accumulated pressure force, that arises as a result of the small-scale flow through the rough topography. Subtracting the long-wave equations in (62), from the unaveraged equation (60), and using the small-scale divergence constraint from (59), we obtain the determining equations for the small-scale component of the leading-order momentum, $\widetilde{H^0u^0}$, and for the small-scale structure of the second order height, H^2. With the $\widetilde{\phi} = \phi - \overline{\phi}$ denoting the small-scale component of some field ϕ, we find the

Balanced small-scale flow equations for rough topography

$$\partial_t\widetilde{H^0u^0} + \mathrm{div}_x(H^0u^0\otimes u^0) + H^0\nabla_x H^2 \,=\, -\widetilde{H^0}\nabla_\chi H^1\,, \tag{64}$$

$$\mathrm{div}_x \widetilde{H^0 u^0} = 0\,. \tag{65}$$

The small-scale flow is driven by the long-wave unbalanced part of the large-scale height gradient, $-\widetilde{H^0}\nabla_\chi H^1$. Since the leading-order momentum is divergence-free on the small scale, the second-order height H^2 assumes the role of a Langrangian multiplier responsible for guaranteeing compliance with this constraint. The nonlinear momentum advection term, $\mathrm{div}_x\left(H^0 u^0 \otimes u^0\right)$, may be rewritten in terms of the large and small-scale momentum components using

$$\begin{aligned} H^0 u^0(t,x,\chi) &= \overline{H^0 u^0}(t,\chi) + \widetilde{H^0 u^0}(t,x,\chi) \\ u^0 &= \frac{1}{H^0}\left(\overline{H^0 u^0} + \widetilde{H^0 u^0}\right) \\ \overline{\widetilde{H^0 u^0}} &\equiv 0 \end{aligned} \quad . \tag{66}$$

Equations (62), (63) and (64), (65) are the shallow-water analog to the single time scale – multiple length scale, variable density low Mach number flow equations derived first in [7], and in the shallow water context in [9].

If the topography does not depend on x, the previous equations may be combined to obtain wave equation with spacially varying signal speed for H^1,

$$\partial_t^2 H^1 - \mathrm{div}_\chi\left((C - B(\chi))\nabla_\chi H^1\right) = 0\,, \tag{67}$$

where $C = B + H^0 \equiv \mathrm{const.}$

5 Conclusions

Exploring a number of multiscale shallow water regimes using formal multiple-scales asymptotics, we find that there is a range of different possibilities for scale interactions in these flows.

- **Balanced flows** For flows that are free of surface waves, a multiscale topography can mediate scale interactions through two mechanisms:

  1. **Weakly nonlinear regime** The large-scale accumulation of net pressure forces on the topography as induced by small-scale flow fluctuations drives the large-scale balanced flow. In turn, the large-scale height gradients that ensure compliance with the large-scale divergence constraint produce small-scale forces when acting on the topographical fluctuations, thereby driving the small-scale flow.
  2. **Strongly nonlinear regime** Here we find the quasi-steady balanced large-scale flow to follow a Darcy-type equation, yet with the homogenized net forces induced not by viscosity or friction, but rather by the accumulation of small-scale nonlinear momentum fluxes. The small-scale flow in turn is driven by the large-scale mean height gradients, and its detailed structure is

determined by the dynamics of vorticity as it gets stretched and compressed when the small-scale flow passes over the variable topography.

- **Long waves passing over multiscale topography** The interaction mechanism across scales is here similar to that found for the weakly nonlinear balanced flow regime, yet the large-scale flow now involves non-balanced free surface waves.

We understand the present paper as a point of departure for further work providing, on the one hand, asymptotic limit cornerstones against which to measure the performance of general shallow water numerical flow solvers. On the other hand, some of the regimes discussed here are of sufficient practical interest to warrant further work on the rigorous mathematical justification of the model equations derived here through merely formal asymptotic arguments. Another direction of research that we intent to pursue concerns viscous regularizations and rigorous justification of the formal analyses presented here.

Acknowledgements
D. Bresch The has been supported by the French "Agence Nationale de la Recherche" project MathOcean (Grant ANR-08-BLAN-0301-01). R. Klein thanks Deutsche Forschungsgemeinschaft for their continued support under grant KL 611/14.

References

1. P. AILLIOT, E. FRÉNOD and V. MONBET, Modeling the Coastal Ocean over a Time Period of Several Weeks. http://hal.archives-ouvertes.fr/hal-00202638/en/.
2. P. AILLIOT, E. FRÉNOD and V. MONBET, Long term object drift forecast in the ocean with tide and wind. *Multiscale Modelling and Simulation*, **5**(2):514–531, 2006.
3. D. BRESCH and D. GÉRARD-VARET, On some homogenization problems from shallow water theory. *Applied Math. Letters*, **20**(5):505–510,2007.
4. E. FRÉNOD, A. MOUTON and E. SONNENDRÜCKER, Two-scale numerical simulation of the weakly compressible 1D isentropic Euler equations. *Numerishe Mathematik*, **108**(2):263–293, 2007.
5. H.P. GREENSPAN, *The Theory of Rotating Fluids*, Cambridge University Press, London, 1968.
6. M. HILLAIRET, Propagation of density-oscillations in solutions to the barotropic compressible Navier-Stokes system. *J. Math. Fluid Mech.*, **9**(3): 343–376, 2007.
7. R. KLEIN, Semi-Implicit Extension of a Godunov-Type Scheme Based on Low Mach Number Asymptotics I: One-dimensional Flow, J. Comput. Phys., **121**, 213–237, 1995.
8. R. KLEIN, Multiple Scales Asymptotics for Atmospheric Flows. In *Proceedings of the 4th European Conference on Mathematics, Stockholm*, 2004.
9. O. LE MAÎTRE, J. LEVIN, M. ISKANDARANI, O.M. KNIO, A Multiple Pressure Splitting of the Shallow–Water Equations, J. Comput. Phys., **166**, 116–151, 2001.
10. P.-L. LIONS and N. MASMOUDI, Homogenization of the Euler system in a 2D porous medium. *J. Math. Pures Appl.*, **84**:1–20, 2005.
11. C. LUCAS, Cosine Effect on Shallow Water Equations and Mathematical Properties. *Quarterly of Applied Mathematics*, **67**(2): 283–310, 2009.
12. A.J. MAJDA, R. KLEIN, Systematic Multi-Scale Models for the Tropics, J. Atmosph. Sci., **60**, 393–408, 2003.
13. F. MARCHE, Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects. *European J. Mech. B / Fluids*, **26**(1): 49–63, 2007.

14. D. Serre, Oscillations non linéaires des systèmes hyperboliques : méthodes et résultats qualitatifs. *Ann. Inst. N. Poincaré Anal. Non Linéaire*, **8**(3–4): 351–417, 1991.
15. D. Serre, Ocillations non linéaires de haute fréquence ; dim = 1. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XII (Paris, 1991-1993)*, volume 302 of *Pitman Res. Notes Math. Ser*, pp 190–210. Longman Sci. Tech., Harlow, 1994.